JHEP HYPER VERSION

Received: *April 7, 1999*, Accepted: *June 16, 1999*

An alternative regularization scheme for the AdS/CFT correspondence

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Abstract: An alternative prescription of the AdS/CFT correspondence is proposed using a regularization procedure, which involves a "smeared" Dirichlet boundary condition. We present the procedure for the example of the scalar field.

Keywords: Renormalization Regularization and Renormalons, Field Theories in Higher Dimensions, $1/N$ Expansion, Conformal and W Symmetry.

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It has been stated in most papers on this subject that the correspondence between a field theory on anti-de Sitter space (AdS) and a conformal field theory (CFT) on its boundary is formally described by the formula [1, 2]

$$e^{-I_{AdS}[\phi]} = \left\langle \exp \int d^d x\, \phi_0(x) \mathcal{O}(x) \right\rangle , \tag{1}$$

where the action I_{AdS} is calculated on-shell for a field configuration ϕ satisfying a Dirichlet condition on the AdS boundary and the boundary value ϕ_0 couples as a current to the conformal field $\mathcal{O}$ living on the AdS boundary. Thus, the formula eq. 1 enables one to calculate correlation functions of the field $\mathcal{O}$ in the boundary conformal field theory. There are two subtle points, which this formal description does not address. First, a Dirichlet boundary value problem in general is not well defined for anti-de Sitter space, since a generic field does not propagate to the boundary. This point has been addressed by demanding an asymptotic boundary behaviour of the AdS fields [2] (asymptotic prescription), or by considering the theory with a cut-off boundary, $x_0 = \epsilon$, where $x_0 = 0$ represents the true AdS boundary (ϵ-prescription). Secondly, the AdS action, $I[\phi_0]$, (specifically the free part) is divergent and must be covariantly regularized and renormalized. There are, of course, several possibilities of regularizing the theory, each of which has its advantages. The ϵ-prescription naturally regularizes the action (i.e. counterterms and the finite result can be read off directly and are independent of each other), whereas in the asymptotic prescription the counterterms will contribute to the finite result. Using these two approaches, the non-contact terms of CFT two-, three- and four-point functions of various fields have been calculated (see [3] for a recent comprehensive list of references). Moreover, some work has focused on obtaining a finite action by interpreting the correlators as distributions [4] and regularizing the action by adding boundary counter terms [5, 6]. Two loop corrections to Super Yang Mills correlators have been calculated in [7] and contribute only to the contact terms.

The purpose of this letter is to suggest an alternative regularization scheme, which is closely related to the ϵ-prescription. Introducing a parameter μ, which somewhat "smears" the validity of the Dirichlet boundary condition at $x_0 = \epsilon$, we shall suggest that it is natural to couple the limits $\mu \to 0$ and $\epsilon \to 0$ with μ being of order ϵ^2. We shall demonstrate the procedure for the free scalar field and show the correctness of the finite result, which is the non-contact term of the CFT two-point function. The new feature of our scheme appears when considering the contact term. Coupling the limits $\epsilon \to 0$ and $\mu \to 0$, we obtain only one such term, which can be made zero in certain cases by appropriately choosing the ratio μ/ϵ^2. We would like to emphasize that it is not our aim to obtain regularized CFT correlation functions.

We shall use the conventional representation of anti-de Sitter space by the space $x^i \in \mathbb{R}$, $(i = 1, \ldots, d)$, $x^0 > 0$ with the metric

$$ds^2 = (x^0)^{-2}\, dx^\mu\, dx^\mu\,. \tag{2}$$

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Let us consider the example of a scalar field [8], which satisfies the equation of motion

$$\left(\nabla^2 - m^2\right)\phi(x) = \left[x_0^2\partial_\mu\partial_\mu - x_0(d-1)\partial_0 - m^2\right]\phi(x) = 0\,. \tag{3}$$

A solution of eq. (3) can be written in the form

$$\phi(x) = \int d^dy\, K(x,\mathbf{y})\phi_\epsilon(\mathbf{y})\,,$$

where $\phi_\epsilon(\mathbf{y})$ is some boundary field and the bulk-boundary kernel is given by

$$K(x,\mathbf{y}) = \int \frac{d^dk}{(2\pi)^d}\left(\frac{x_0}{\epsilon}\right)^{d/2}\frac{K_\alpha(kx_0)}{K_\alpha(k\epsilon)}e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})-\mu k^2}\,. \tag{4}$$

Here, K_α is a modified Bessel function (Mac Donald function) and α is related to the mass parameter by

$$\alpha = \sqrt{\frac{d^2}{4}+m^2}\,.$$

In addition to previous work [8], we have introduced the regulating factor $e^{-\mu k^2}$ in order to make the integral well defined for all values of x_0 and $\mathbf{x}-\mathbf{y}$. In the limit $\mu\to 0$, eq. (4) reduces to the standard Dirichlet kernel.

The CFT two-point function is determined by the boundary normal derivative of the kernel eq. 4, which is given by

$$\partial_0 K(x,\mathbf{y})|_\epsilon = \frac{1}{\epsilon}\int\frac{d^dk}{(2\pi)^d}\left\{\frac{d}{2}-\alpha+k\frac{\partial}{\partial k}\ln\left[(k\epsilon)^\alpha K_\alpha(k\epsilon)\right]\right\}e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})-\mu k^2}\,. \tag{5}$$

Consider first the case of non-coincident points $\mathbf{x}$ and $\mathbf{y}$. As without regularization, we use the expansion

$$\begin{aligned} z^\alpha K_\alpha(z) = 2^{\alpha-1}\Gamma(\alpha)\Bigg[1+\sum_{j=1}^\infty\frac{1}{j!(1-\alpha)_j}\left(\frac{z}{2}\right)^{2j} - \\ -\frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)}\left(\frac{z}{2}\right)^{2\alpha}\sum_{j=0}^\infty\frac{1}{j!(1+\alpha)_j}\left(\frac{z}{2}\right)^{2j}\Bigg], \end{aligned} \tag{6}$$

where we used the notation $(a)_j = \Gamma(a+j)/\Gamma(a)$. Proceeding to expand the logarithm in eq. (5) one obtains

$$z\frac{\partial}{\partial z}\ln\left[z^\alpha K_\alpha(z)\right] = \frac{z^2}{2(1-\alpha)}+\cdots-\frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)}2^{1-2\alpha}\alpha z^{2\alpha}+\cdots\,, \tag{7}$$

where the first set of dots indicates analytic terms of order z^{2n} $(n>1)$ and the second set higher order non-analytic terms. Substituting eq. (7) into eq. (5) we recognize integrals of the type

$$\int \frac{d^d k}{(2\pi)^d} k^\beta e^{-i\mathbf{k}\cdot\mathbf{x}-\mu k^2} = \frac{|\mathbf{x}|^{1-d/2}}{(2\pi)^{d/2}} \int\limits_0^\infty dk\, k^{d/2+\beta} J_{d/2-1}(k|\mathbf{x}|)\, e^{-\mu k^2} = $$
$$= \frac{\Gamma\left(d+\beta/2\right)}{2^d \pi^{d/2}\Gamma\left(d/2\right)\mu^{\frac{d+\beta}{2}}} \Phi\left(\frac{d+\beta}{2}; \frac{d}{2}; -\frac{|\mathbf{x}|^2}{4\mu}\right). \tag{8}$$

Here, $\Phi(a,c,z)$ is the degenerate hypergeometric function. For $\mathbf{x} \neq 0$ we can take the $\mu \to 0$ limit and replace Φ with the leading term of its asymptotic expansion [9], which yields

$$\int \frac{d^d k}{(2\pi)^d} k^\beta e^{-i\mathbf{k}\cdot\mathbf{x}-\mu k^2} \overset{\mu\to 0}{=} \frac{2^\beta \Gamma(d+\beta/2)}{\pi^{d/2}\Gamma(-\beta/2)} \frac{1}{|\mathbf{x}|^{d+\beta}}\,. \tag{9}$$

We notice that for the analytic terms, $\beta = 2n$, the gamma function in the denominator diverges, so the integral becomes zero. The same holds for the subleading terms of the asymptotic expansion. Hence, the analytic terms do not contribute to the finite distance two-point function. On the other hand, for the leading non-analytic term, $\beta = 2\alpha$, we obtain (still for $\mathbf{x} \neq \mathbf{y}$)

$$\partial_0 K(x,\mathbf{y})|_\epsilon \overset{\mu\to 0}{=} 2\alpha c_\alpha \frac{\epsilon^{2\alpha-1}}{|\mathbf{x}-\mathbf{y}|^{2\Delta}}\,, \tag{10}$$

where $\Delta = d/2+\alpha$ and $c_\alpha = \Gamma(\Delta)/(\pi^{d/2}\Gamma(\alpha))$. The higher order non-analytic terms in eq. (7) contribute with higher powers of ϵ and can be neglected in the $\epsilon \to 0$ limit. Hence, we find agreement with previous results [8]. A nice and simple check of this procedure is provided by the conformally coupled scalar field, $\alpha = 1/2$, where the calculation can be done exactly.

Let us now turn to the local terms in the two-point function. In this case, it is more useful not to carry out the expansion 7, since all its terms would contribute because of $\Phi(a,c,0) = 1$. Consider instead the expression 5 directly with $\mathbf{x} = \mathbf{y}$. Using the identity

$$z\frac{\partial}{\partial z}\ln[z^\alpha K_\alpha(z)] = -z\frac{K_{\alpha-1}(z)}{K_\alpha(z)}$$

and changing the integration variable to $\mathbf{s} = \sqrt{\mu}\,\mathbf{k}$ we find

$$\partial_0 K(x,\mathbf{x})|_\epsilon = \frac{1}{\epsilon(4\pi\mu)^{d/2}}\left[\frac{d}{2} - \alpha - \frac{2a}{\Gamma(d/2)}\int ds\, s^d e^{-s^2}\frac{K_{\alpha-1}(as)}{K_\alpha(as)}\right], \tag{11}$$

where the parameter a denotes the ratio $a = \epsilon/\sqrt{\mu}$. The s-integral is well defined for all positive a, although in general not elementary. However, we know that it represents some positive number. Hence, we obtain

$$\partial_0 K(x,\mathbf{x})|_\epsilon = -\frac{\gamma}{\epsilon(4\pi\mu)^{d/2}}\,, \tag{12}$$

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where γ is a regularization dependent parameter satisfying

$$\alpha - \frac{d}{2} < \gamma < \infty \,.$$

Moreover, if $2\alpha < d$, i.e. for $-d^2 < 4m^2 < 0$, one could determine a such that $\gamma = 0$. For example, in the conformally coupled case, $\alpha = 1/2$, one finds

$$a_{1/2} = \frac{(d-1)\Gamma(d/2)}{\Gamma(d+1/2)} \,.$$

The contribution 12 to the two-point function can be compensated by adding a local counterterm

$$I_c = -\frac{\gamma}{2} \int d^d x \, \epsilon^{-d} \big[\phi_\epsilon(\mathbf{x})\big]^2 \tag{13}$$

to the AdS action of a scalar field. However, one must notice that this only eliminates the contact term, but does not regularize the two-point function.

In conclusion, we have presented in this letter an alternative regularization scheme for the AdS/CFT correspondence. Our calculation shows that in general there is a contact contribution to the two-point function, but it can be compensated by adding a covariant local surface term to the AdS action or made zero by appropriately choosing the ratio μ/ϵ^2. Alternatively, it might serve as a regulator for the CFT two-point function and thus eliminate the need to add counterterms. We think that this is an interesting topic for further study.

Acknowledgments

This work was supported in part by a grant from NSERC. W.M. is grateful to Simon Fraser University for financial support.

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